

# Explicit asymptotic velocity of the boundary between particles and antiparticles

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## Abstract

On the real line initially there are infinite number of particles on the positive half-line., each having one of  $K$  negative velocities  $v_1^{(+)}, \dots, v_K^{(+)}$ . Similarly, there are infinite number of antiparticles on the negative half-line, each having one of  $L$  positive velocities  $v_1^{(-)}, \dots, v_L^{(-)}$ . Each particle moves with constant speed, initially prescribed to it. When particle and antiparticle collide, they both disappear. It is the only interaction in the system. We find explicitly the large time asymptotics of  $\beta(t)$  - the coordinate of the last collision before  $t$  between particle and antiparticle.

Keywords: phase boundary dynamics, random walks in cones, piece-wise linear dynamical systems, one instrument market.

## 1 Introduction

We consider one-dimensional dynamical model of the boundary between two phases (particles and antiparticles, bears and bulls) where the boundary moves due to reaction (annihilation, transaction) of pairs of particles of different phases.

Assume that at time  $t = 0$  infinite number of  $(+)$ -particles and  $(-)$ -particles are situated correspondingly on  $R_+$  and  $R_-$  and have one-point correlation functions

$$f_+(x, v) = \sum_{i=1}^K \rho_i^{(+)}(x) \delta(v - v_i^{(+)}), \quad f_-(x, v) = \sum_{j=1}^L \rho_j^{(-)}(x) \delta(v - v_j^{(-)})$$

Moreover for any  $i, j$

$$v_i^{(+)} < 0, \quad v_j^{(-)} > 0$$

that is two phases move towards each other. Particles of the same phase do not see each other and move freely with the velocities prescribed initially. The only interaction in the system is the following. When two particles of different phases find themselves at the same point they immediately disappear (annihilate). It follows that the phases stay separated, and one might call any point in-between them the phase boundary (for example it could be the point of the last collision). Thus the boundary trajectory  $\beta(t)$  is a random piece-wise constant function of time.

One of the possible interpretations is the simplest model of one instrument (for example, a stock) market. Particle initially at  $x(0) \in R_+$  is the seller who wants to sell his stock for

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the price  $x(0)$ , which is higher than the existing price  $\beta(0)$ . There are  $K$  groups of sellers characterized by their activity to move towards more realistic price. Similarly the  $(-)$ -particles are buyers who would like to buy a stock for the price lower than  $\beta(t)$ . When seller and buyer meet each other, the transaction occurs and both leave the market.

The main result of the paper is the explicit formula for the asymptotic velocity of the boundary as the function of  $2(K + L)$  parameters - densities and initial velocities. It appears to be continuous but at some hypersurface some first derivatives in the parameters do not exist. This kind of phase transition has very clear interpretation: the particles with smaller activities (velocities) cease to participate in the boundary movement - they are always behind the boundary, that is do not influence the market price  $\beta(t)$ . In this paper we consider only the case of constant densities  $\rho_i^{(+)}, \rho_i^{(-)}$ , that is the period of very small volatility in the market. This simplification allows us to get explicit formulas. In [3] much simpler case  $K = L = 1$  was considered, however with non-constant densities and random dynamics.

Other one-dimensional models (hardly related to ours) of the boundary movement see in [9, 10].

Main technical tool of the proof may seem surprising (and may be of its own interest) - we reduce this infinite particle problem to the study of a special random walk of one particle in the orthant  $R_+^N$  with  $N = KL$ . The asymptotic behavior of this random walk is studied using the correspondence between random walks in  $R_+^N$  and dynamical systems introduced in [1].

The organization of the paper is the following. In section 2 we give exact formulation of the model and of the main result. In section 3 we introduce the correspondence between infinite particle process, random walks and dynamical systems. In sections 4 and 5 we give the proofs.

## 2 Model and the main result

**Initial conditions** At time  $t = 0$  on the real axis there is a random configuration of particles, consisting of  $(+)$ -particles and  $(-)$ -particles.  $(+)$ -particles and  $(-)$ -particles differ also by the type: denote  $I_+ = \{1, 2, \dots, K\}$  the set of types of  $(+)$ -particles, and  $I_- = \{1, 2, \dots, L\}$  - the set of types of  $(-)$ -particles. Let

$$0 < x_{1,k} = x_{1,k}(0) < \dots < x_{j,k} = x_{j,k}(0) < \dots \quad (1)$$

be the initial configuration of particles of type  $k \in I_+$ , and

$$\dots < y_{j,i} = y_{j,i}(0) < \dots < y_{1,i} = y_{1,i}(0) < 0 \quad (2)$$

be the initial configuration of particles of type  $i \in I_-$ , where the second index is the type of the particle in the configuration. Thus all  $(+)$ -particles are situated on  $R_+$  and all  $(-)$ -particles on  $R_-$ . Distances between neighbor particles of the same type are denoted by

$$\begin{aligned} x_{j,k} - x_{j-1,k} &= u_{j,k}^{(+)}, & k \in I_+, & j = 1, 2, \dots \\ y_{j-1,i} - y_{j,i} &= u_{j,i}^{(-)}, & i \in I_-, & j = 1, 2, \dots \end{aligned} \quad (3)$$

where we put  $x_{0,k} = y_{0,i} = 0$ . The random configurations corresponding to the particles of different types are assumed to be independent. The random distances between neighbor particles of the same type are also assumed to be independent, and moreover identically distributed, that is random variables  $u_{j,i}^{(-)}, u_{j,k}^{(+)}$  are independent and their distribution depends only on the upper and second lower indices. Our technical assumption is that all these distributions are

absolutely continuous and have finite means. Denote  $\mu_i^{(-)} = Eu_{j,i}^{(-)}$ ,  $\rho_i^{(-)} = \left(\mu_i^{(-)}\right)^{-1}$ ,  $i \in I_-$ ,  $\mu_k^{(+)} = Eu_{j,k}^{(+)}$ ,  $\rho_k^{(+)} = \left(\mu_k^{(+)}\right)^{-1}$ ,  $k \in I_+$ .

**Dynamics** We assume that all (+)-particles of the type  $k \in I_+$  move in the left direction with the same constant speed  $v_k^{(+)}$ , where  $v_1^{(+)} < v_2^{(+)} < \dots < v_K^{(+)} < 0$ . The (-)-particles of type  $i \in I_-$  move in the right direction with the same constant speed  $v_i^{(-)}$ , where  $v_1^{(-)} > v_2^{(-)} > \dots > v_L^{(-)} > 0$ . If at some time  $t$  a (+)-particle and a (-)-particle are at the same point (we call this a collision or annihilation event), then both disappear. Collisions between particles of different phases is the only interaction, otherwise they do not see each other. Thus, for example, at time  $t$  the  $j$ -th particle of type  $k \in I_+$  could be at the point

$$x_{j,k}(t) = x_{j,k}(0) + v_k^{(+)}t$$

if it will not collide with some (-)-particle before time  $t$ . Absolute continuity of the distributions of random variables  $u_{j,i}^{(-)}, u_{j,k}^{(+)}$  guaranties that the events when more than two particles collide, have zero probability.

We denote this infinite particle process  $\mathbf{D}(t)$ .

We define the boundary  $\beta(t)$  between plus and minus phases to be the coordinate of the last collision which occurred at some time  $t' < t$ . For  $t = 0$  we put  $\beta(0) = 0$ . Thus the trajectories of the random process  $\beta(t)$  are piecewise constant functions, we shall assume them continuous from the left.

**Main result** For any pair  $(J_-, J_+)$  of subsets,  $J_- \subseteq I_-$ ,  $J_+ \subseteq I_+$ , define the numbers

$$V(J_-, J_+) = \frac{\sum_{i \in J_-} v_i^{(-)} \rho_i^{(-)} + \sum_{k \in J_+} v_k^{(+)} \rho_k^{(+)}}{\sum_{i \in J_-} \rho_i^{(-)} + \sum_{k \in J_+} \rho_k^{(+)}} , V = V(I_-, I_+) \quad (4)$$

The following condition is assumed

$$\{V(J_-, J_+) : J_- \neq \emptyset, J_+ \neq \emptyset\} \cap \{v_1^{(-)}, \dots, v_L^{(-)}, v_1^{(+)}, \dots, v_K^{(+)}\} = \emptyset. \quad (5)$$

If the limit  $W = \lim_{t \rightarrow \infty} \frac{\beta(t)}{t}$  exists a.e., we call it the asymptotic speed of the boundary. Our main result is the explicit formula for  $W$ .

**Theorem 1** *The asymptotic velocity of the boundary exists and is equal to*

$$W = V(\{1, \dots, L_1\}, \{1, \dots, K_1\})$$

where

$$L_1 = \max \left\{ l \in \{1, \dots, L\} : v_l^{(-)} > V(\{1, \dots, l\}, I_+) \right\}, \quad (6)$$

$$K_1 = \max \left\{ k \in \{1, \dots, K\} : v_k^{(+)} < V(I_-, \{1, \dots, k\}) \right\}. \quad (7)$$

Note that the definition of  $L_1$  and  $K_1$  is not ambiguous because  $v_1^{(-)} > V(\{1\}, I_+)$  and  $v_1^{(+)} < V(I_-, \{1\})$ .

Now we will explain this result in more detail. As  $v_K^{(+)} < 0 < v_L^{(-)}$ , there can be 3 possible orderings of the numbers  $v_L^{(-)}, v_K^{(+)}, V$ :

1.  $v_K^{(+)} < V < v_L^{(-)}$ . In this case

$$K_1 = K, \quad L_1 = L, \quad W = V$$

2. If  $v_K^{(+)} > V$  then  $V < 0$  and  $K_1 < K$ ,  $L_1 = L$ . Moreover

$$W = V(\{1, \dots, L\}, \{1, \dots, K_1\}) = \min_{k \in I_+} V(\{1, \dots, L\}, \{1, \dots, k\}) < V < 0$$

3. If  $v_L^{(-)} < V$  then  $V > 0$  and  $K_1 = K$ ,  $L_1 < L$ . Moreover

$$W = V(\{1, \dots, L_1\}, I_+) = \max_{l \in I_-} V(\{1, \dots, l\}, I_+) > V > 0$$

The item 1 is evident. The items 2 and 3 will be explained in section 6.2.

**Another scaling** Normally the minimal difference between consecutive prices (a tick) is very small. Moreover one customer can have many units of the commodity. That is why it is natural to consider the scaled densities

$$\rho_j^{(+),\epsilon} = \epsilon^{-1} \rho_j^{(+)}, \quad \rho_j^{(-),\epsilon} = \epsilon^{-1} \rho_j^{(-)}$$

for some fixed constants  $\rho_j^{(+)}, \rho_j^{(-)}$ . Then the phase boundary trajectory  $\beta^{(\epsilon)}(t)$  will depend on  $\epsilon$ . The results will look even more natural. Namely, it follows from the main theorem, that for any  $t > 0$  there exists the following limit in probability

$$\beta(t) = \lim_{\epsilon \rightarrow 0} \beta^{(\epsilon)}(t)$$

that is the limiting boundary trajectory.

**Example of phase transition** The case  $K = L = 1$ , that is when the activities of (+)-particles are the same (and similarly for (-)-particles), is very simple. There is no phase transition in this case. The boundary velocity

$$W = \frac{v_1^{(+)} \rho_1^{(+)} + v_1^{(-)} \rho_1^{(-)}}{\rho_1^{(+)} + \rho_1^{(-)}} \quad (8)$$

depends analytically on the activities and densities. This is very easy to prove because the  $n$ -th collision time is given by the simple formula

$$t_n = \frac{x_n^{(+)}(0) - x_n^{(-)}(0)}{-v_1^{(+)} + v_1^{(-)}} \quad (9)$$

and  $n$ -th collision point is given by

$$x_n^{(+)}(0) + t_n v_1^{(+)} = x_n^{(-)}(0) + t_n v_1^{(-)}. \quad (10)$$

More complicated situation was considered in [3]. There the movement of (+)-particles has random jumps in both directions with constant drift  $v_1^{(+)} \neq 0$  (and similarly for (-)-particles). In [3] *the order of particles of the same type can be changed* with time. There are no such simple formulas as (9) and (10) in this case. The result is however the same as in (8).

The phase transition appears already in case when  $K = 2$ ,  $L = 1$  and moreover the  $(-)$ -particles stand still, that is  $v_1^{(-)} = 0$ . Denote  $\rho_1^{(-)} = \rho_0$ ,  $v_i^{(+)} = v_i$ ,  $\rho_i^{(+)} = \rho_i$ ,  $i = 1, 2$ . Consider the function

$$V_1(v_1, \rho_1) = \frac{\rho_1 v_1}{\rho_0 + \rho_1}.$$

It is the asymptotic speed of the boundary in the system where there is no  $(+)$ -particles of type 2 at all.

Then the asymptotic velocity is the function

$$W = V(v_1, v_2, \rho_1, \rho_2) = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_0 + \rho_1 + \rho_2}$$

if  $v_2 < V_1$  and

$$W = V_1(v_1, \rho_1) = \frac{\rho_1 v_1}{\rho_0 + \rho_1}$$

if  $v_2 > V_1$ . We see that at the point  $v_2 = V_1$  the function  $W$  is not differentiable in  $v_2$ .

### 3 Random walk and dynamical system in $R_+^N$

**Associated random walk** One can consider the phase boundary as a special kind of server where the customers (particles) arrive in pairs and are immediately served. However the situation is more involved than in standard queuing theory, because the server moves, and correlation between its movement and arrivals is sufficiently complicated. That is why this analogy does not help much. However we describe the crucial correspondence between random walks in  $R_+^N$  and the infinite particle problem defined above, that allows to get the solution.

Denote  $b_i^{(-)}(t)$  ( $b_k^{(+)}(t)$ ) the coordinate of the extreme right (left), and still existing at time  $t$ , that is not annihilated at some time  $t' < t$ ,  $(-)$ -particle of type  $i \in I_-$  ( $(+)$ -particle of type  $k \in I_+$ ). Define the distances  $d_{i,k}(t) = b_k^{(+)}(t) - b_i^{(-)}(t) \geq 0$ ,  $i \in I_-$ ,  $k \in I_+$ . The trajectories of the random processes  $b_i^{(-)}(t)$ ,  $b_k^{(+)}(t)$ ,  $d_{i,k}(t)$  are assumed left continuous. Consider the random process  $D(t) = (d_{i,k}(t), (i, k) \in I) \in R_+^N$ , where  $N = KL$ .

Denote  $\mathcal{D} \in R_+^N$  the state space of  $D(t)$ . Note that the distances  $d_{i,k}(t)$ , for any  $t$ , satisfy the following conservation laws

$$d_{i,k}(t) + d_{n,m}(t) = d_{i,m}(t) + d_{n,k}(t)$$

where  $i \neq n$  and  $k \neq m$ . That is why the state space  $\mathcal{D}$  can be given as the set of non-negative solutions of the system of  $(L-1)(K-1)$  linear equations

$$d_{1,1} + d_{n,m} = d_{1,m} + d_{n,1}$$

where  $n, m \neq 1$ . It follows that the dimension of  $\mathcal{D}$  equals  $K + L - 1$ . However it is convenient to speak about random walk in  $R_+^N$ , taking into account that only subset of dimension  $K + L - 1$  is visited by the random walk.

Now we describe the trajectories  $D(t)$  in more detail. The coordinates  $d_{i,k}(t)$  decrease linearly with the speeds  $v_i^{(-)} - v_k^{(+)}$  correspondingly until one of the coordinates  $d_{i,k}(t)$  becomes zero. Let  $d_{i,k}(t_0) = 0$  at some time  $t_0$ . This means that  $(-)$ -particle of type  $i$  collided with  $(+)$ -particle of type  $k$ . Let them have numbers  $j$  and  $l$  correspondingly. Then the components

of  $D(t)$  become:

$$\begin{aligned} d_{i,k}(t_0 + 0) &= u_{j+1,i}^{(-)} + u_{l+1,k}^{(+)} \\ d_{i,m}(t_0 + 0) - d_{i,m}(t_0) &= u_{j+1,i}^{(-)}, \quad m \neq k \\ d_{n,k}(t_0 + 0) - d_{n,k}(t_0) &= u_{l+1,k}^{(+)}, \quad n \neq i \end{aligned}$$

and other components will not change at all, that is do not have jumps.

Note that the increments of the coordinates  $d_{n,m}(t_0 + 0) - d_{n,m}(t_0)$  at the jump time do not depend on the history of the process before time  $t_0$ , as the random variables  $u_{j,i}^{(-)}(u_{j,k}^{(+)})$  are independent and equally distributed for fixed type. It follows that  $D(t)$  is a Markov process. However that this continuous time Markov process has singular transition probabilities (due to partly deterministic movement). This fact however does not prevent us from using the techniques from [1] where random walks in  $Z_+^N$  were considered.

**Ergodic case** We call the process  $D(t)$  ergodic, if there exists a neighborhood  $A$  of zero, such that the mean value  $E\tau_x$  of the first hitting time  $\tau_x$  of  $A$  from the point  $x$  is finite for any  $x \in \mathcal{D}$ . In the ergodic case the correspondence between boundary movement and random walks is completely described by the following theorem.

**Theorem 2** *Two following two conditions are equivalent:*

- 1) *The process  $D(t)$  is ergodic;*    2)  $v_K^{(+)} < V < v_L^{(-)}$ .

All other cases of boundary movement correspond to non-ergodic random walks. Even more, we will see that in all other cases the process  $D(t)$  is transient. Condition (5), which excludes the set of parameters of zero measure, excludes in fact null recurrent cases.

To understand the corresponding random walk dynamics introduce a new family of processes.

**Faces** Let  $\Lambda \subseteq I = I_- \times I_+$ . The face of  $R_+^N$  associated with  $\Lambda$  is defined as

$$\mathcal{B}(\Lambda) = \{x \in R_+^N : x_{i,k} > 0, (i, k) \in \Lambda, x_{i,k} = 0, (i, k) \in \bar{\Lambda}\} \subseteq R_+^N \quad (11)$$

If  $\Lambda = \emptyset$ , then  $\mathcal{B}(\Lambda) = \{0\}$ . For shortness, instead of  $\mathcal{B}(\Lambda)$  we will sometimes write  $\Lambda$ . However, one should note that the inclusion like  $\Lambda \subset \Lambda_1$  is ALWAYS understood for subsets of  $I$ , not for the faces themselves.

Define the following set of “appropriate” faces  $\mathcal{G} = \{\Lambda : \bar{\Lambda} = J_- \times J_+, J_- \subseteq I_-, J_+ \subseteq I_+\}$ .

**Lemma 1**

$$\mathcal{D} = \bigcup_{\Lambda_0 \in \mathcal{G}} (\mathcal{D} \cap \Lambda_0).$$

The proof will be given in Section 5.5. This lemma explains why in the study of the process  $D(t)$  we can consider only “appropriate” faces.

## Induced process

One can define a family  $\mathbf{D}(t; J_-, J_+)$  of infinite particle processes, where  $J_- \subseteq I_-$ ,  $J_+ \subseteq I_+$ . The process  $\mathbf{D}(t; J_-, J_+)$  is the process  $\mathbf{D}(t)$  with  $\rho_j^{(+)} = 0, j \notin J_+$  and  $\rho_j^{(-)} = 0, j \notin J_-$ . All other parameters (that is the densities and velocities) are the same as for  $\mathbf{D}(t)$ . Note that these processes are in general defined on different probability spaces. Obviously  $\mathbf{D}(t; I_-, I_+) = \mathbf{D}(t)$ .

Similarly to  $\mathbf{D}(t)$ , the processes  $\mathbf{D}(t; J_-, J_+)$  have associated random walks  $D(t; J_-, J_+)$  in  $R_+^{N_1}$  with  $N_1 = |J_-||J_+|$ . Usefulness of these processes is that they describe all possible types of asymptotic behavior of the main process  $D(t)$ .

Consider a face  $\Lambda \in \mathcal{G}$ , i.e., such face that its complement  $\bar{\Lambda} = J_- \times J_+$  where  $J_- \subseteq I_-$  and  $J_+ \subseteq I_+$ . The process  $D_\Lambda(t) = D(t; J_-, J_+) = (d_{i,k}^\Lambda(t), (i, k) \in \bar{\Lambda})$  will be called an **induced process**, associated with  $\Lambda$ . The coordinates  $d_{i,k}^\Lambda(t)$  are defined in the same way as  $d_{i,k}(t) = d_{i,k}^\Lambda(t)$ , where  $\bar{\Lambda} = \{\emptyset\}$ . The state space of this process is  $\mathcal{D}^{\bar{\Lambda}} = \mathcal{D}(R^{|\bar{\Lambda}|})$ , where  $|\bar{\Lambda}| = |J_-||J_+|$ . Face  $\Lambda$  is called **ergodic** if the induced process  $D_\Lambda(t)$  is ergodic.

## Induced vectors

Introduce the plane

$$\mathcal{R}(\Lambda) = \{x \in R^N : x_{i,k} = 0, (i, k) \in \bar{\Lambda}\} \subseteq R^N$$

**Lemma 2** *Let  $\Lambda$  be ergodic with  $\bar{\Lambda} = J_- \times J_+$ , and  $D_y(t)$  be the process  $D(t)$  with the initial point  $y \in \mathcal{B}(\Lambda)$ . Then there exists vector  $v^\Lambda \in \mathcal{R}(\Lambda)$  such that for any  $y \in \mathcal{B}(\Lambda)$   $t \geq 0$ , such that  $y + v^\Lambda t \in \mathcal{B}(\Lambda)$ , we have as  $M \rightarrow \infty$*

$$\frac{D_{yM}(tM)}{M} \rightarrow y + v^\Lambda t$$

This vector  $v^\Lambda$  will be called the **induced vector** for the ergodic face  $\Lambda$ . We will see other properties of the induced vector below.

## Non-ergodic faces

Let  $\Lambda$  be the face which is not ergodic (non-ergodic face). Ergodic face  $\Lambda_1$ :  $\Lambda_1 \supset \Lambda$  will be called outgoing for  $\Lambda$ , if  $v_{i,k}^{\Lambda_1} > 0$  for  $(i, k) \in \Lambda_1 \setminus \Lambda$ . Let  $\mathcal{E}(\Lambda)$  be the set of outgoing faces for the non-ergodic face  $\Lambda$ .

**Lemma 3** *The set  $\mathcal{E}(\Lambda)$  contains the minimal element  $\Lambda_1$  in the sense that for any  $\Lambda_2 \in \mathcal{E}(\Lambda)$  we have  $\Lambda_2 \supseteq \Lambda_1$ .*

This lemma will be proved in section 5.2.

## Dynamical system

We define now the piece-wise constant **vector field**  $v(x)$  in  $\mathcal{D}$ , consisting of induced vectors, as follows:  $v(x) = v^\Lambda$  if  $x$  belongs to ergodic face  $\Lambda$ , and  $v(x) = v^{\Lambda_1}$  if  $x$  belongs to non-ergodic face  $\Lambda$ , where  $\Lambda_1$  is the minimal element of  $\mathcal{E}(\Lambda)$ . Let  $U^t$  be the **dynamical system** corresponding to this vector field.

It follows that the trajectories  $\Gamma_x = \Gamma_x(t)$  of the dynamical system are piecewise linear. Moreover, if the trajectory hits a non-ergodic face, it leaves it immediately. It goes with constant speed along an ergodic face until it reaches its boundary.

We call the ergodic face  $\Lambda = \mathcal{L}$  final, if either  $\mathcal{L} = \emptyset$  or all coordinates of the induced vector  $v^{\mathcal{L}}$  are positive. The central statement is that the dynamical system hits the final face, stays on it forever and goes along it to infinity, if  $\mathcal{L} \neq \emptyset$ .

The following theorem, together with theorem 2, is parallel to theorem 1. That is in all 3 cases of theorem 1, theorems 2 and 3 describe the properties of the corresponding random walks in the orthant.

### Theorem 3

1. If  $D(t)$  is ergodic then the origin is the fixed point of the dynamical system  $U^t$ . Moreover, all trajectories of the dynamical system  $U^t$  hit 0.
2. Assume  $v_K^{(+)} > V$ . Then the process  $D(t)$  is transient and there exists a unique ergodic final face  $\mathcal{L}$ , such that  $v_{i,k}^{\mathcal{L}} > 0$  for  $(i, k) \in \mathcal{L}$ . This face is

$$\mathcal{L}(L, K_1) = \{(i, k) : i = 1, \dots, L, k = K_1 + 1, \dots, K\}$$

where  $K_1$  is defined by (7). Moreover, all trajectories of the dynamical system  $U^t$  hit  $\mathcal{L}(L, K_1)$  and stay there forever.

3. Assume  $v_L^{(-)} < V$ . Then the process  $D(t)$  is transient and there exists a unique ergodic final face  $\mathcal{L}$ , such that  $v_{i,k}^{\mathcal{L}} > 0$  for  $(i, k) \in \mathcal{L}$ . This face is

$$\mathcal{L}(L_1, K) = \{(i, k) : i = L_1 + 1, \dots, L, k = 1, \dots, K\}$$

where  $L_1$  is defined by (6). Moreover, all trajectories of the dynamical system  $U^t$  hit  $\mathcal{L}(L_1, K)$  and stay there forever.

4. For any initial point  $x$  the trajectory  $\Gamma_x(t)$  has finite number of transitions from one face to another, until it reaches  $\{0\}$  or one of the final faces.

This theorem will be proved in section 5.3.

**Simple examples of random walks and dynamical systems** If  $K = L = 1$  the process  $D(t)$  is a random process on  $R_+$ . It is deterministic on  $R_+ \setminus \{0\}$  - it moves with constant velocity  $v^{(+)} - v^{(-)}$  towards the origin. When it reaches 0 at time  $t$ , it jumps backwards

$$D(t+0) = \eta$$

where  $\eta$  has the same distribution as  $u_1^{(+)} + u_1^{(-)}$ . The dynamical system coincides with  $D(t)$  inside  $R_+$ , and has the origin as its fixed point.

If  $L = 1, K = 2$  and moreover  $v_1^{(-)} = 0$  then the state space of the process is  $R_+^2 = \{(d_{11}, d_{12})\}$ . Inside the quarter plane the process is deterministic and moves with velocity  $(v_1^{(+)}, v_2^{(+)})$ . From any point  $x$  of the boundary  $d_{12} = 0$  it jumps to the random point  $x + \eta_1$ , and from any point of the boundary  $d_{11} = 0$  it jumps to the point  $x + \eta_2$ , where  $\eta_1, \eta_2$  have the same distributions as  $(u_{j,1}^{(-)}, u_{j,1}^{(-)} + u_{j,2}^{(+)})$  and  $(u_{j,1}^{(-)} + u_{j,1}^{(+)}, u_{j,1}^{(-)})$  correspondingly. The classification results for random walks in  $Z_+^2$  can be easily transferred to this case; the dynamical system is deterministic and has negative components of the velocity inside  $R_+^2$ . When it hits one of the axes it moves along it. The velocity is always negative along the first axis, however along second axis it can be either negative or positive. This is the phase transition we described above. Correspondingly the origin is the fixed point in the first case, and has positive value of the vector field along the second axis, in the second case.



## 4 Collisions

**Basic process** Now we come back to our infinite particle process  $\mathbf{D}(t)$ . The collision of particles of the types  $i \in I_-$ ,  $k \in I_+$  we shall call the collision of type  $(i, k)$ . Denote

$$\nu_{i,k}(T) = \#\{t : d_{i,k}(t) = 0, t \in [0, T]\}$$

the number of collisions of type  $(i, k)$  on the time interval  $[0, T]$ .

**Lemma 4** *If the process  $D(t)$  is ergodic, then the following positive limits exist a.s.*

$$\pi_{i,k} = \lim_{T \rightarrow \infty} \frac{\nu_{i,k}(T)}{T} > 0, (i, k) \in I \quad (12)$$

and satisfy the following system of linear equations

$$v_i^{(-)} - v_k^{(+)} = \sum_{(n,m) \in I_- \times I_+} (\delta(n, i) \mu_i^{(-)} + \delta(m, k) \mu_k^{(+)}) \pi_{n,m}, (i, k) \in I \quad (13)$$

Proof. Remind that the collisions can be presented as follows. If  $d_{i,k}(t_0) = 0$ , then for any  $n, m$

$$d_{n,m}(t_0 + 0) - d_{n,m}(t_0) = \delta(n, i) u_{j+1,i}^{(-)} + \delta(m, k) u_{l+1,k}^{(+)}$$

where  $\delta(n, i) = 1$  for  $n = i$  and  $\delta(n, i) = 0$  for  $n \neq i$ . Note that the proof of (12) is similar to the proof of the corresponding assertion in [2]. For large  $t$  we have

$$d_{i,k}(t) = -(v_i^{(-)} - v_k^{(+)})t + \sum_{(n,m) \in I_- \times I_+} (\delta(n, i) \mu_i^{(-)} + \delta(m, k) \mu_k^{(+)}) \nu_{n,m}(t) + o(t)$$

Note that this is exact equality, if instead of  $\mu_i^{(-)}$  and  $\mu_k^{(+)}$  we take random distances between particles. By the law of large numbers and by (12), the system (13) follows.

We shall need below the following new notation. The equations (13) can be rewritten in the new variables  $\pi_i^{(-)}, \pi_k^{(+)}$  as follows

$$v_i^{(-)} - v_k^{(+)} = \pi_i^{(-)} \mu_i^{(-)} + \pi_k^{(+)} \mu_k^{(+)}$$

where

$$\pi_i^{(-)} = \sum_{m=1}^K \pi_{i,m}, \pi_k^{(+)} = \sum_{n=1}^L \pi_{n,k}$$

Obviously the following balance equation holds

$$\sum_{i=1}^L \pi_i^{(-)} = \sum_{k=1}^K \pi_k^{(+)} = \sum_{i=1}^L \sum_{k=1}^K \pi_{i,k}$$

Rewrite the system (13) in a more convenient form, using the variables  $r_i^{(-)} = \pi_i^{(-)} \mu_i^{(-)}$ ,  $r_k^{(+)} = \pi_k^{(+)} \mu_k^{(+)}$ . Then

$$\begin{aligned} v_i^{(-)} - v_k^{(+)} &= r_i^{(-)} + r_k^{(+)}, (i, k) \in I \\ \sum_{i=1}^L r_i^{(-)} \rho_i^{(-)} &= \sum_{k=1}^K r_k^{(+)} \rho_k^{(+)} \end{aligned}$$

It follows that for all  $(i, k) \in I$

$$v_i^{(-)} - r_i^{(-)} = r_k^{(+)} + v_k^{(+)}$$

Introduce the variable  $w = v_i^{(-)} - r_i^{(-)} = r_k^{(+)} + v_k^{(+)}$ . We get the following system of equations with respect to the variables  $r_i^{(-)}, r_k^{(+)}, w$ :

$$\begin{aligned} v_i^{(-)} - r_i^{(-)} &= w, & i \in I_- \\ r_k^{(+)} + v_k^{(+)} &= w, & k \in I_+ \\ \sum_{i=1}^L r_i^{(-)} \rho_i^{(-)} &= \sum_{k=1}^K r_k^{(+)} \rho_k^{(+)} \end{aligned} \tag{14}$$

It is easy to see that this system has the unique solution

$$r_i^{(-)} = v_i^{(-)} - w, \quad r_k^{(+)} = -v_k^{(+)} + w, \quad w = V \tag{15}$$

where  $V$  is defined by (4). If  $D(t)$  is ergodic, then by lemma 4 we have  $r_i^{(-)}, r_k^{(+)} > 0$  for any  $i \in I_-, k \in I_+$ .

**Lemma 5** *Let the process  $D(t)$  be ergodic. Then*

- 1).  $v_K^{(+)} < V < v_L^{(-)}$ .
- 2). *The speed of the boundary  $W = V$ .*

Proof. 1). If  $D(t)$  is ergodic, then by lemma 4  $\pi_i^{(-)} > 0$  and  $\pi_k^{(+)} > 0$  for all  $i \in I_-, k \in I_+$ . So, by (15) we have

$$r_i^{(-)} = v_i^{(-)} - V > 0, \quad r_k^{(+)} = -v_k^{(+)} + V > 0$$

2). Let  $\nu_i^{(-)}(T)$  be the number of particles of type  $i \in I_-$ , which had collisions during time  $T$ . Then

$$\sum_{j=1}^{\nu_i^{(-)}(T)} u_{j,i}^{(-)}$$

is the initial coordinate of the particle of type  $i \in I$ , which was the last annihilated among the particle of this type. Let  $T_i$  be the annihilation time of this particle. Then

$$\frac{\beta(T_i + 0) + \sum_{j=1}^{\nu_i^{(-)}(T)} u_{j,i}^{(-)}}{T_i} = v_i^{(-)}$$

Rewrite this expression as follows

$$\frac{\beta(T_i + 0) - \beta(T) + \beta(T) + \sum_{j=1}^{\nu_i^{(-)}(T)} u_{j,i}^{(-)}}{T} = \frac{T_i}{T} v_i^{(-)}$$

It follows that

$$\frac{\beta(T)}{T} = \frac{T_i}{T} v_i^{(-)} - \frac{\sum_{j=1}^{\nu_i^{(-)}(T)} u_{j,i}^{(-)}}{T} + \frac{\beta(T) - \beta(T_i + 0)}{T}$$

By lemma 4 and the strong law of large numbers

$$\frac{\sum_{j=1}^{\nu_i^{(-)}(T)} u_{j,i}^{(-)}}{T} = \frac{\nu_i^{(-)}(T)}{T} \frac{\sum_{j=1}^{\nu_i^{(-)}(T)} u_{j,i}^{(-)}}{\nu_i^{(-)}(T)} \rightarrow \pi_i^{(-)} \mu_i^{(-)} = r_i^{(-)}, \text{ a.e.}$$

as  $T \rightarrow \infty$ . At the same time ergodicity of the process  $D(t)$  gives that as  $T \rightarrow \infty$

$$\frac{T - T_i}{T} \rightarrow 0, \quad \frac{\beta(T) - \beta(T_i + 0)}{T} \rightarrow 0, \quad a.e.$$

Thus for any  $i \in I_-$  a.e.

$$\lim_{T \rightarrow \infty} \frac{\beta(T)}{T} = v_i^{(-)} - r_i^{(-)} = V$$

Similarly one can prove that for all  $k \in I_+$

$$\lim_{T \rightarrow \infty} \frac{\beta(T)}{T} = v_k^{(+)} + r_k^{(+)}$$

It follows from equations (14) and (15) that the boundary velocity is defined by (4). Lemma is proved.

**Induced process** Consider the faces  $\Lambda$  such that  $\bar{\Lambda} = J_- \times J_+$ , where  $J_- \subseteq I_-$  and  $J_+ \subseteq I_+$ . Let

$$\nu_{i,k}^\Lambda(T) = \#\{t : d_{i,k}^\Lambda(t) = 0, t \in [0, T]\}$$

be the number of collisions of type  $(i, k)$  on the time interval  $[0, T]$  in the process  $\mathbf{D}(t; J_-, J_+)$ .

The following lemma is quite similar to lemma 4.

**Lemma 6** *If the process  $D_\Lambda(t)$  is ergodic then the following a.e. limits exist and are positive for all pairs  $(i, k) \in \bar{\Lambda}$*

$$\pi_{i,k}^\Lambda = \lim_{T \rightarrow \infty} \frac{\nu_{i,k}^\Lambda(T)}{T} > 0 \quad (16)$$

*They satisfy the following system of linear equations*

$$v_i^{(-)} - v_k^{(+)} = \sum_{(n,m) \in \bar{\Lambda}} (\delta(n, i) \mu_i^{(-)} + \delta(m, k) \mu_k^{(+)}) \pi_{n,m}^\Lambda, \quad (i, k) \in \bar{\Lambda} \quad (17)$$

Introduce the following notation

$$\begin{aligned} \pi_i^{(\Lambda, -)} &= \sum_{k \in J_+} \pi_{i,k}^\Lambda, \quad i \in J_- \\ \pi_k^{(\Lambda, +)} &= \sum_{i \in J_-} \pi_{i,k}^\Lambda, \quad k \in J_+ \\ r_i^{(\Lambda, -)} &= \mu_i^{(-)} \pi_i^{(\Lambda, -)}, \quad i \in J_- \\ r_k^{(\Lambda, +)} &= \mu_k^{(+)} \pi_k^{(\Lambda, +)}, \quad k \in J_+ \end{aligned}$$

For  $\Lambda = \emptyset, \bar{\Lambda} = I_- \times I_+$  we have  $\pi_i^{(\Lambda, -)} = \pi_i^{(-)}, \pi_k^{(\Lambda, +)} = \pi_k^{(+)}$  and  $r_i^{(\Lambda, -)} = r_i^{(-)}, r_k^{(\Lambda, +)} = r_k^{(+)}$ .

Due to (17) for  $(i, k) \in \bar{\Lambda}$  we have

$$v_i^{(-)} - v_k^{(+)} = \sum_{(n,m) \in \bar{\Lambda}} (\delta(n, i) \mu_i^{(-)} + \delta(m, k) \mu_k^{(+)}) \pi_{n,m}^\Lambda = \mu_i^{(-)} \pi_i^{(\Lambda, -)} + \mu_k^{(+)} \pi_k^{(\Lambda, +)} = r_i^{(\Lambda, -)} + r_k^{(\Lambda, +)} \quad (18)$$

It follows that  $v_i^{(-)} - r_i^{(\Lambda, -)} = r_k^{(\Lambda, +)} + v_k^{(+)}$  for all  $(i, k) \in \bar{\Lambda}$ . Put  $w^\Lambda = v_i^{(-)} - r_i^{(\Lambda, -)} = r_k^{(\Lambda, +)} + v_k^{(+)}$ . In this way we have obtained the following system of linear equations (similar the system (14)) with respect to variables  $r_i^{(\Lambda, -)}, r_k^{(\Lambda, +)}, w^\Lambda$ :

$$\begin{aligned}
v_i^{(-)} - r_i^{(\Lambda, -)} &= w^\Lambda, & i \in I_- \\
r_k^{(\Lambda, +)} + v_k^{(+)} &= w^\Lambda, & k \in I_+ \\
\sum_{i \in J_-} \rho_i^{(-)} r_i^{(\Lambda, -)} &= \sum_{k \in J_+} \rho_k^{(+)} r_k^{(\Lambda, +)}
\end{aligned} \tag{19}$$

As previously, this system has the unique solution

$$r_i^{(\Lambda, -)} = v_i^{(-)} - w^\Lambda, \quad r_k^{(\Lambda, +)} = -v_k^{(+)} + w^\Lambda, \quad w^\Lambda = V^\Lambda = V(J_-, J_+) \tag{20}$$

For any process  $\mathbf{D}(t; J_-, J_+)$  or for the corresponding induced process  $D_\Lambda(t)$  (see Section 3), we also define the boundary  $\beta^\Lambda(t)$  as the coordinate of the last collision  $(i, k) \in \bar{\Lambda}$  before  $t$ . Let us assume that  $\beta^\Lambda(0) = 0$ . The trajectories of the random process  $\beta^\Lambda(t)$  are also piece-wise constant, we shall assume them left continuous. The following lemma is completely analogous to lemma 5.

**Lemma 7** *Let  $\bar{\Lambda} = J_- \times J_+ = \{i_l, \dots, i_1\} \times \{k_1, \dots, k_m\}$ , where  $i_l > \dots > i_1$  and  $k_1 < \dots < k_m$ , and let  $\Lambda$  be an ergodic face. Then*

- 1).  $v_{i_l}^{(-)} > V^\Lambda = V(J_-, J_+)$  and  $v_{k_m}^{(+)} < V^\Lambda = V(J_-, J_+)$
- 2). *The boundary velocity for the process  $\mathbf{D}(t; J_-, J_+)$  (or for the corresponding  $D_\Lambda(t)$ ) equals (with the a.e. limit)*

$$\lim_{t \rightarrow \infty} \frac{\beta^\Lambda(t)}{t} = V^\Lambda = V(J_-, J_+)$$

Note that  $V^\Lambda = V$  for  $\Lambda = \emptyset$ .

**Lemma 8** *For any ergodic face  $\Lambda$  ( $\bar{\Lambda} = J_- \times J_+$ ) the vector  $v^\Lambda \in \mathcal{R}(\Lambda)$  with the coordinates equal to*

$$v_{i,k}^\Lambda = -v_i^{(-)} + v_k^{(+)} + 1(i \in J_-) \mu_i^{(-)} \pi_i^{(\Lambda, -)} + 1(k \in J_+) \mu_k^{(+)} \pi_k^{(\Lambda, +)}, \quad (i, k) \in \Lambda \tag{21}$$

*is the induced vector in the sense of lemma 2.*

This is quite similar to lemma 2.2, page 143 of [KMR] and lemma 4.3.2, page 87 of [4].

It follows from (21) and (20), that the coordinates of the induced vector are given by

$$v_{i,k}^\Lambda = -v_i^{(-)} + V^\Lambda, \quad (i, k) \in \Lambda, i \notin J_-, k \in J_+ \tag{22}$$

$$v_{i,k}^\Lambda = v_k^{(+)} - V^\Lambda, \quad (i, k) \in \Lambda, i \in J_-, k \notin J_+ \tag{23}$$

$$v_{i,k}^\Lambda = -v_i^{(-)} + v_k^{(+)}, \quad (i, k) \in \Lambda, i \notin J_-, k \notin J_+ \tag{24}$$

$$v_{i,k}^\Lambda = 0, \quad (i, k) \in \bar{\Lambda}$$

Note that by condition (5) for all induced vectors  $v_{i,k}^\Lambda \neq 0$  if  $(i, k) \in \Lambda$ .

Intuitive interpretation of this formula is the following. For example the inequality  $v_{i,k}^\Lambda = -v_i^{(-)} + V^\Lambda < 0$ ,  $(i, k) \in \Lambda$ ,  $i \notin J_-$ ,  $k \in J_+$  means that  $(-)$ -particles of type  $i \in I_-$  overtake the boundary which moves with velocity  $V^\Lambda$ . In the contrary case,  $v_{i,k}^\Lambda = -v_i^{(-)} + V^\Lambda > 0$ , that is  $(-)$ -particles of type  $i \in I_-$  fall behind the boundary.

## 5 Proofs

### 5.1 Proof of theorem 2

The implication  $1 \Rightarrow 2$  has been proved in lemma 5. Now we prove that 2) implies 1). We will use the method of Lyapounov functions to prove ergodicity. Define the Lyapounov function

$$f(y) = \sum_{(i,k) \in I} p_{i,k} y_{i,k} = (p, y)$$

where vector  $p$  with coordinates  $p_{i,k} > 0$  will be defined below. One has to verify the following condition: there exists  $\delta > 0$  such that for any ergodic face  $\Lambda$ ,  $\Lambda \neq \{0\}$ ,

$$f(y + v^\Lambda) - f(y) = (p, v^\Lambda) < -\delta$$

where  $v^\Lambda$  is the induced vector corresponding to the face  $\Lambda$ , see [4].

The system (13) can be written in the matrix form

$$v = A\pi \tag{25}$$

where  $A$  is the  $N \times N$  matrix

$$A = \{a_{(i,k),(n,m)} = \delta(n, i)\mu_i^{(-)} + \delta(m, k)\mu_k^{(+)}\}, \tag{26}$$

with the elements indexed by  $(i, k) \in I$ , and the vector

$$v = \{v_{(i,k)} = v_i^{(-)} - v_k^{(+)}, (i, k) \in I\} \tag{27}$$

It is easy to see that the coordinates of the vector  $A\pi$  are equal to

$$(A\pi)_{i,k} = \mu_i^{(-)}\pi_i^{(-)} + \mu_k^{(+)}\pi_k^{(+)}$$

If the assumption 2) of the theorem holds, then the system of equations (14) has a positive solution, that is,  $r_i^{(-)}, r_k^{(+)} > 0$ . One can choose positive  $p_{i,k}$  so that the following condition holds

$$\pi_i^{(-)} = \sum_{m=1}^K p_{i,m}, \quad \pi_k^{(+)} = \sum_{n=1}^L p_{n,k}$$

where  $\pi_i^{(-)} = \rho_i^{(-)}r_i^{(-)}$  and  $\pi_k^{(+)} = \rho_k^{(+)}r_k^{(+)}$ . For example, one can put

$$p_{i,m} = C^{-1}\pi_i^{(-)}\pi_k^{(+)}$$

where

$$C = \sum_{i=1}^L \pi_i^{(-)} = \sum_{k=1}^K \pi_k^{(+)}$$

Let the vector  $p$  have coordinates  $p_{i,k}$ . Then  $p$  satisfies the system (25), that is  $v = Ap$ .

For ergodic face  $\Lambda$  define the vector  $\pi^\Lambda$  with coordinates  $\pi_{i,k}^\Lambda$ , where  $\pi_{i,k}^\Lambda$  for  $(i, k) \in \bar{\Lambda}$  are defined in (16) and we put  $\pi_{i,k}^\Lambda = 0$  for  $(i, k) \in \Lambda$ . It follows from (18) and (21), that the induced vector can be written as

$$v^\Lambda = -v + A\pi^\Lambda \tag{28}$$

with the matrix  $A$  and the vector  $v$  defined in (26) and (27). By (28) we have

$$v^\Lambda = -v + A\pi^\Lambda = -A(p - \pi^\Lambda)$$

As the vector  $A(p - \pi^\Lambda)$  belongs to the face  $\Lambda$  and  $Pr_\Lambda \pi^\Lambda = 0$ , then

$$f(y + v^\Lambda) - f(y) = (p, v^\Lambda) = -(p, A(p - \pi^\Lambda)) = -(p - \pi^\Lambda, A(p - \pi^\Lambda))$$

Note that the matrix  $A$  in (25) is a nonnegative operator. In fact, for any vector  $y = (y_{i,j}) \in R^N$

$$(Ay, y) = \sum_{i,k} (\mu_i^{(-)} y_i^{(-)} + \mu_k^{(+)} y_k^{(+)}) y_{i,k} = \sum_{i=1}^L \mu_i^{(-)} \left( y_i^{(-)} \right)^2 + \sum_{k=1}^K \mu_k^{(+)} \left( y_k^{(+)} \right)^2 \geq 0 \quad (29)$$

where

$$y_i^{(-)} = \sum_{m=1}^K y_{i,m}, \quad y_k^{(+)} = \sum_{n=1}^L y_{n,k}$$

Let for definiteness  $\bar{\Lambda} = J_- \times J_+$ . By formula (29)

$$\begin{aligned} -(p - \pi^\Lambda, A(p - \pi^\Lambda)) &= - \sum_{i=1}^L \mu_i^{(-)} \left( \pi_i^{(-)} - \pi_i^{(\Lambda, -)} \right)^2 - \sum_{k=1}^K \mu_k^{(+)} \left( \pi_k^{(+)} - \pi_k^{(\Lambda, +)} \right)^2 \\ &< - \sum_{i \notin J_-} \mu_i^{(-)} \left( \pi_i^{(-)} \right)^2 - \sum_{k \notin J_+} \mu_k^{(+)} \left( \pi_k^{(+)} \right)^2 < 0 \end{aligned}$$

as  $\pi_i^{(-)}, \pi_k^{(+)} > 0$ ,  $\pi_i^{(\Lambda, -)} = 0$  for  $i \notin J_-$ ,  $\pi_k^{(\Lambda, +)} = 0$  if  $k \notin J_+$ . As the number of faces is finite, one can always choose  $\delta > 0$ , so that

$$f(y + v^\Lambda) - f(y) = -(p - \pi^\Lambda, A(p - \pi^\Lambda)) < -\delta$$

The theorem is proved.

## 5.2 Proof of lemma 3

For any non-ergodic face  $\Lambda$  with  $\bar{\Lambda} = J_- \times J_+ = \{i_1, \dots, i_l\} \times \{m_1, \dots, m_k\}$ , where  $i_1 < \dots < i_l$  and  $m_1 < \dots < m_k$ , define

$$\begin{aligned} q &= \max \left\{ n \in \{1, \dots, l\} : v_{i_n}^{(-)} > V(\{i_1, \dots, i_n\}, \{m_1, \dots, m_k\}) \right\}, \\ r &= \max \left\{ j \in \{1, \dots, k\} : v_{m_j}^{(+)} < V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_j\}) \right\}. \end{aligned} \quad (30)$$

This definition is correct because always  $v_{i_1}^{(-)} > V(\{i_1\}, \{m_1, \dots, m_k\})$  and  $v_{m_1}^{(+)} < V(\{i_1, \dots, i_l\}, \{m_1\})$ .

Introduce the face  $\Lambda_1$  such that  $\bar{\Lambda}_1 = \{i_1, \dots, i_q\} \times \{m_1, \dots, m_r\}$ . If  $r = k$ ,  $q = l$ , then  $v_{m_k}^{(+)} < V(J_-, J_+) < v_{i_l}^{(-)}$  and  $\Lambda_1 = \Lambda$ . By theorem 2 the induced process  $D_\Lambda(t)$  is ergodic and the face  $\Lambda$  is ergodic.

So there can be two possible cases:

- If  $r < k$ ,  $q = l$ , then  $\bar{\Lambda}_1 = \{i_1, \dots, i_l\} \times \{m_1, \dots, m_r\}$ ,  $v_{m_k}^{(+)} > V(J_-, J_+)$  and  $V(J_-, J_+) < 0$ .

- If  $r = k$ ,  $q < l$ , then  $\overline{\Lambda}_1 = \{i_1, \dots, i_q\} \times \{m_1, \dots, m_k\}$ ,  $v_{i_l}^{(-)} < V(J_-, J_+)$  and  $V(J_-, J_+) > 0$ .

By construction we have  $\Lambda_1 \supset \Lambda$ .

We show that  $\Lambda_1$  is the minimal ergodic outgoing face for  $\Lambda$ . Consider the first case, namely  $r < k$ ,  $q = l$ . The second one is quite similar. Because of  $v_{m_r}^{(+)} < V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_r\}) < v_{i_l}^{(-)}$  we can apply theorem 2 and so the induced process  $D_{\Lambda_1}(t)$  is ergodic. This gives ergodicity of the face  $\Lambda_1$ .

By formula (23) for all  $(i_n, m_j) \in \Lambda_1 \setminus \Lambda = \{i_1, \dots, i_l\} \times \{m_{r+1}, \dots, m_k\}$

$$v_{i_n, m_j}^{\Lambda_1} = v_{m_j}^{(+)} - V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_r\})$$

and by formula (30)

$$v_{m_j}^{(+)} > V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_r, m_{r+1}, \dots, m_j\})$$

It follows from lemma 13 that

$$V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_r\}) < V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_r, m_{r+1}, \dots, m_j\})$$

Thus, we get  $v_{i_n, m_j}^{\Lambda_1} > 0$  for all  $(i_n, m_j) \in \Lambda_1 \setminus \Lambda$ . It means that the face  $\Lambda_1$  is outgoing for  $\Lambda$ .

To finish the proof of lemma 3 it is sufficient to show that the constructed face  $\Lambda_1$  is the minimal outgoing face for  $\Lambda$ . We give the proof by contradiction. Let there exist an ergodic outgoing (for  $\Lambda$ ) face  $\Lambda_0 \supset \Lambda$  such that  $\Lambda_0 \neq \Lambda_1$  and  $\Lambda_1 \cap \Lambda_0 \neq \Lambda_1$ . Put

$$\overline{\Lambda}_0 = J_-^0 \times J_+^0 \subset \overline{\Lambda} = \{i_1, \dots, i_l\} \times \{m_1, \dots, m_k\}$$

By (22)-(24) the coordinates  $v_{i,k}^{\Lambda_0}$  of the induced vector  $v^{\Lambda_0}$  are given for  $(i, k) \in \Lambda_0 \setminus \Lambda$  as follows

$$\begin{aligned} v_{i,k}^{\Lambda_0} &= -v_i^{(-)} + V(J_-^0, J_+^0), (i, k) \in (J_- \setminus J_-^0) \times J_+^0, \\ v_{i,k}^{\Lambda_0} &= v_k^{(+)} - V(J_-^0, J_+^0), (i, k) \in J_-^0 \times (J_+ \setminus J_+^0), \\ v_{i,k}^{\Lambda_0} &= -v_i^{(-)} + v_k^{(+)}, (i, k) \in J_- \setminus J_-^0 \times J_+ \setminus J_+^0 \end{aligned}$$

As the face  $\Lambda_0$  is outgoing we must have  $v_{i,k}^{\Lambda_0} > 0$  for all  $(i, k) \in \Lambda_0 \setminus \Lambda$ . Thus, the only two situations are possible:  $\overline{\Lambda}_0 = J_-^0 \times \{m_1, \dots, m_k\}$  or  $\overline{\Lambda}_0 = \{i_1, \dots, i_l\} \times J_+^0$ . In the first case we have

$$v_{i,j}^{\Lambda_0} = -v_i^{(-)} + V(J_-^0, \{m_1, \dots, m_k\}) > 0, (i, j) \in (J_- \setminus J_-^0) \times \{m_1, \dots, m_k\}$$

and so  $V(J_-^0, \{m_1, \dots, m_k\}) > 0$ . But then  $V(J_-, J_+) > 0$  and this contradicts the assumption  $V(J_-, J_+) < 0$ .

So  $\overline{\Lambda}_0 = \{i_1, \dots, i_l\} \times J_+^0$ . Show that  $J_+^0 = \{m_1, \dots, m_r\}$ .

Let  $J_+^0 \neq \{m_1, \dots, m_r\}$  and there is  $j \in \{m_1, \dots, m_r\}$  such that  $j \notin J_+^0$ . Then by lemma 13

$$v_{i,j}^{\Lambda_0} = v_j^{(+)} - V(J_-^0, J_+^0) < 0$$

and, hence, the face  $\Lambda_0$  can not be outgoing for  $\Lambda$ . If  $\{m_1, \dots, m_r\} \subset J_+^0$  there exists some point  $(i_n, m_j) \in \Lambda_0 \setminus \Lambda$ , where  $j \in \{r+1, \dots, k\}$  and by (30)

$$v_{m_j}^{(+)} > V(\{i_1, \dots, i_l\}, \{m_1, \dots, m_r, m_{r+1}, \dots, m_j\})$$

It follows from theorem 2 that the induced process  $D_{\Lambda_0}(t)$  is non-ergodic and, hence, the face  $\Lambda_0$  is also non-ergodic. This contradicts the assumption on ergodicity of the face  $\Lambda_0$ . So  $J_+^0 = \{m_1, \dots, m_r\}$ . Lemma is proved.

### 5.3 Proof of theorem 3

The first goal of this subsection is to study trajectories  $\Gamma(t)$  of the dynamical system  $U_t$ . After that, using the obtained knowledge about behavior of  $\Gamma(t)$  we shall prove Theorem 3. Let  $\Gamma_x(t)$  be the trajectory of the dynamical system, starting in the point  $\Gamma_x(0) = x \in R_+^N$ .

According to the definition of  $U_t$  any trajectory  $\Gamma_x(t)$ ,  $t \geq 0$ , visits some sequence of faces. In general, this sequence depends on the initial point  $x$  and contains ergodic and non ergodic faces. It is very complicated to give a precise list of all faces visited by the concrete trajectory started from a given point  $x$ . Our idea is to find a common *finite subsequence*  $\Lambda_1, \Lambda_2, \dots, \Lambda_n$  of ergodic faces in the order they are visited by any trajectory. We find this subsequence together with the time moments  $t_1, t_2, \dots, t_n$ , where  $t_k$  is the first time the trajectory enters the closure of  $\Lambda_k$ . Moreover, it will follow from our proof that the intervals  $t_k - t_{k-1}$  are finite, the dimensions of the ergodic faces in this sequence decrease and any trajectory, after hitting the closure of some face in this sequence, will never leave this closure.

**Proposition 4** *There exists a monotone sequence of faces*

$$\Lambda_1 \supset \Lambda_2 \supset \dots \supset \Lambda_r \supset \dots \supset \Lambda_n, \quad \dim F_i > \dim F_{i+1},$$

*and a sequence of time moments*

$$t_1 \leq t_2 \leq \dots \leq t_r \leq \dots \leq t_n < +\infty,$$

*depending on  $x$ , and having the following property*

$$\Gamma_x(t) \in F_r \quad \forall t \geq t_r,$$

where  $F_r = cl(\Lambda_r)$  denotes the closure of  $\Lambda_r$  in  $R_+^N$ . Moreover, the sequence  $\Lambda_1, \Lambda_2, \dots, \Lambda_n$  depends only on the parameters of the model (that is on the velocities and densities), but the sequence of time moments  $t_1, t_2, \dots, t_n$  depends also on the initial point  $x$  of the trajectory  $\Gamma_x(t)$ . Thus any trajectory will hit the final set  $F_{fin} = F_n$  in finite time.

The proof of Proposition 4 will be given at the end of this subsection.

First, we shall present here some algorithm for constructing the sequence  $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ . By Lemma 1 we can consider only faces  $\Lambda$ , such that  $\bar{\Lambda} = J^{(-)} \times J^{(+)}$ . Algorithm consists of several number of steps and constructs a sequence  $\bar{\Lambda}_1, \bar{\Lambda}_2, \dots$ ,

$$\bar{\Lambda}_p = J_p^{(-)} \times J_p^{(+)} = \{(l, k) \mid l \in J_p^{(-)}, k \in J_p^{(+)}\}. \quad (31)$$

In fact it constructs a sequence  $\left\{ (J_p^{(-)}, J_p^{(+)}) \right\}_{p=1}^n$ . We prefer here to use notation

$$(J_p^{(-)}, J_p^{(+)}) = T_p = (J_p^{(-)} \mid J_p^{(+)})$$

and to call  $T_p$  a group consisting of particle types listed in  $J_p^{(-)}, J_p^{(+)}$ .

Notation  $V^{T_i}$  has the same meaning as earlier

$$V^{T_i} = \frac{\sum_{l \in J_i^{(-)}} v_l^{(-)} \rho_l^{(-)} + \sum_{k \in J_i^{(+)}} v_k^{(+)} \rho_k^{(+)}}{\sum_{l \in J_i^{(-)}} \rho_l^{(-)} + \sum_{k \in J_i^{(+)}} \rho_k^{(+)}}.$$

*Algorithm:*



1) Put  $T_1 = (1 \mid 1)$  and find  $V^{T_1}$

2a) If  $V^{T_1} < 0$ , compare  $-v_2^{(+)}$  and  $|V^{T_1}|$ .

- If  $-v_2^{(+)} > |V^{T_1}|$ , then  $T_2 = (1 \mid 1, 2)$ .
- If  $-v_2^{(+)} < |V^{T_1}|$ , then  $T_2 = (2, 1 \mid 1)$ .

2b) If  $V^{T_1} > 0$ , compare  $v_2^{(-)}$  and  $V^{T_1}$ .

- If  $v_2^{(-)} > V^{T_1}$ , then  $T_2 = (2, 1 \mid 1)$ .
- If  $v_2^{(-)} < V^{T_1}$ , then  $T_2 = (1 \mid 1, 2)$ .

... Let we have already constructed group

$$T_{r-1} = (b, b-1, \dots, 1 \mid 1, \dots, a-1, a).$$

Find  $V^{T_{r-1}}$ . If  $a < K$  and  $b < L$  hold, then apply the following steps  $r$ -a) and  $r$ -b).

$r$ -a) If  $V^{T_{r-1}} < 0$  and  $a < K$ , compare  $-v_{a+1}^{(+)}$  and  $|V^{T_{r-1}}|$ .

- If  $-v_{a+1}^{(+)} > |V^{T_{r-1}}|$ , then  $T_r = (b, \dots, 1 \mid 1, \dots, a, a+1)$ .
- If  $-v_{a+1}^{(+)} < |V^{T_{r-1}}|$ , then  $T_r = (b+1, b, \dots, 1 \mid 1, \dots, a)$ .

$r$ -b) If  $V^{T_{r-1}} > 0$  and  $b < L$ , we compare  $v_{b+1}^{(-)}$  and  $V^{T_{r-1}}$ .

- If  $v_{b+1}^{(-)} > V^{T_{r-1}}$ , then  $T_r = (b+1, b, \dots, 1 \mid 1, \dots, a)$ .
- If  $v_{b+1}^{(-)} < V^{T_{r-1}}$ , then  $T_r = (b, \dots, 1 \mid 1, \dots, a, a+1)$ .

$r$ -c) If  $a = K$ , and  $b < L$ , we compare  $v_{b+1}^{(-)}$  and  $V^{T_{r-1}}$ .

- If  $v_{b+1}^{(-)} > V^{T_{r-1}}$ , then  $T_r = (b+1, b, \dots, 1 \mid 1, \dots, K)$ .
- If  $v_{b+1}^{(-)} < V^{T_{r-1}}$ , then the algorithm is finished and the group  $T_{r-1} = (b, \dots, 1 \mid 1, \dots, K)$  is declared to be the final group  $T_{fin}$  of the algorithm.

$r$ -d) If  $a < K$ , and  $b = L$ , we compare  $v_{a+1}^{(+)}$  and  $V^{T_{r-1}}$ .

- If  $v_{a+1}^{(+)} < V^{T_{r-1}}$ , then  $T_r = (L, \dots, 1 \mid 1, \dots, a, a+1)$ .
- If  $v_{a+1}^{(+)} > V^{T_{r-1}}$ , then the algorithm is finished and the group  $T_{r-1} = (L, \dots, 1 \mid 1, \dots, a)$  is declared to be the final group  $T_{fin}$  of the algorithm.

$r$ -e) If  $a = K$  and  $b = L$ , then the algorithm is finished and the group  $T_{r-1} = (L, \dots, 1 \mid 1, \dots, K)$  is declared to the final group  $T_{fin}$  of the algorithm.

If the algorithm did not stop at the steps  $r$ -c),  $r$ -d) or  $r$ -e), then the step  $r + 1$  should be fulfilled, etc. It is clear that the algorithm stops after finite number of steps, and as the result we get a final group  $T_{fin}$ , which will have one of the following types

$$(L, \dots, 1 \mid 1, \dots, K), \quad (L, \dots, 1 \mid 1, \dots, K_1), \quad (L_1, \dots, 1 \mid 1, \dots, K), \quad (32)$$

where  $K_1 < K$ ,  $L_1 < L$ .

We need not only the final group, corresponding to the face along which the trajectory escapes to infinity, but also the whole chain

$$T_1 = (1 \mid 1) \rightarrow T_2 \rightarrow T_3 \rightarrow \dots \rightarrow T_{fin}. \quad (33)$$

As it follows from the algorithm, this chain is uniquely defined by the parameters of the model.

Let us remark, that in the algorithm we excluded cases where some of  $V^{T_{r-1}}$  are zero. We will show below (see Remark 10) how to modify the algorithm to take into account these cases as well.

The next lemma is needed for the proof of the theorem 3. It is convenient however to give this proof here, as it is essentially based on the details of the algorithm defined above.

### Lemma 9

1. If  $T_{fin} = (L, \dots, 1 \mid 1, \dots, K)$ , then simultaneously  $v_L^{(-)} > V^{T_{fin}}$  and  $v_K^{(+)} < V^{T_{fin}}$  hold.
2. If  $T_{fin} = (L, \dots, 1 \mid 1, \dots, K_1)$ , where  $K_1 < K$ , then  $V^{T_{fin}} < 0$  and  $v_K^{(+)} > V^{T_{fin}}$ .
3. If  $T_{fin} = (L_1, \dots, 1 \mid 1, \dots, K)$ , where  $L_1 < L$ , then  $V^{T_{fin}} > 0$  and  $v_L^{(-)} < V^{T_{fin}}$ .

*Proof of Lemma 9.* In fact, if  $T_{fin} = (L, \dots, 1 \mid 1, \dots, K_1)$ , where  $K_1 < K$ , then the algorithm stops on some step  $r_0$ -d), and thus, the condition  $v_{K_1+1}^{(+)} > V^{T_{fin}}$  will hold. As  $0 > v_K^{(+)} \geq v_{K_1+1}^{(+)}$ , then we get the proof of the part 2 of the lemma. Part 3 is quite similar.

To prove assertion 1 of the lemma consider the face, previous to the final one.

$$T_{fin} = (L, \dots, 1 \mid 1, \dots, K).$$

Two cases are possible:

$$T_{f-1} = (L, \dots, 1 \mid 1, \dots, K-1) \text{ or } T_{f-1} = (L-1, \dots, 1 \mid 1, \dots, K).$$

Consider the case  $T_{f-1} = (L, \dots, 1 \mid 1, \dots, K-1)$  and the final fragment of the trajectory in the algorithm:

$$(L-1, \dots, 1 \mid 1, \dots, q) \rightarrow (L, \dots, 1 \mid 1, \dots, q) \rightarrow \dots \rightarrow T_{f-1} = (L, \dots, 1 \mid 1, \dots, K-1) \rightarrow T_{fin}.$$

Two cases of the first transition in this chain are possible:

- 1)  $V^{(L-1, \dots, 1 \mid 1, \dots, q)} < 0$  and  $v_{q+1}^{(+)} > V^{(L-1, \dots, 1 \mid 1, \dots, q)}$ .
- 2)  $V^{(L-1, \dots, 1 \mid 1, \dots, q)} > 0$  and  $v_L^{(-)} > V^{(L-1, \dots, 1 \mid 1, \dots, q)}$ .

In both cases one can claim that

$$v_L^{(-)} > V^{(L, \dots, 1 \mid 1, \dots, q)}. \quad (34)$$

To prove this consider both cases separately.

Case 1) As  $v_L^{(-)} > 0$ , then we have  $v_L^{(-)} > V^{(L-1, \dots, 1 | 1, \dots, q)}$ . Thus,  $v_L^{(-)} > V^{(L, \dots, 1 | 1, \dots, q)}$ , as  $V^{(L, \dots, 1 | 1, \dots, q)}$  is the convex linear combination (CLC<sup>1</sup>)  $v_L^{(-)}$  and  $V^{(L-1, \dots, 1 | 1, \dots, q)}$ .

Case 2) Here we assume  $v_L^{(-)} > V^{(L-1, \dots, 1 | 1, \dots, q)}$ . From this, as above, we get that  $v_L^{(-)} > V^{(L, \dots, 1 | 1, \dots, q)}$ .

Thus, the inequality (34) is proved. As  $V^{(L, \dots, 1 | 1, \dots, K)}$  is CLC of  $V^{(L, \dots, 1 | 1, \dots, q)}$  and negative numbers  $v_{q+1}^{(+)}, \dots, v_K^{(+)}$ , then

$$V^{(L, \dots, 1 | 1, \dots, K)} < V^{(L, \dots, 1 | 1, \dots, q)}.$$

Then we have  $V^{(L, \dots, 1 | 1, \dots, K)} < v_L^{(-)}$ .

The latter transition in the chain occurs because  $v_K^{(+)} < V^{(L, \dots, 1 | 1, \dots, K-1)}$ . Then  $v_K^{(+)} < V^{(L, \dots, 1 | 1, \dots, K)}$ , as  $V^{(L, \dots, 1 | 1, \dots, K)}$  is CLC of  $V^{(L, \dots, 1 | 1, \dots, K-1)}$  and  $v_K^{(+)}$ .

This gives the proof.

Let  $a_r$  and  $b_r$  are such that

$$T_r = (b_r, \dots, 1 | 1, \dots, a_r). \quad (35)$$

The numbers  $a_r$  and  $b_r$  are non-decreasing functions of  $r$ . Moreover  $a_r + b_r$  increases by 1 if  $r$  increases by 1. What can be the difference between  $T_{r-1}$  and  $T_r$ ? There can be two cases:

Case  $\Pi_r$ :  $a_r = a_{r-1} + 1, b_r = b_{r-1}$ .

Case  $U_r$ :  $a_r = a_{r-1}, b_r = b_{r-1} + 1$ .

Remind that the face  $B(\Lambda) \in R_+^N$  is defined by the set of pairs of indices  $\Lambda \subseteq I_- \times I_+$ . Namely, to each pair  $(j, k) \in \Lambda$  corresponds positive coordinates  $d_{j,k} > 0$  in the definition (11) of the face  $B(\Lambda)$  and vice-versa. For shortness we say that the face  $B(\Lambda)$  consists of pairs  $(j, k) \in \Lambda$ .

**Proposition 5** *Let the chain (33) be given and case  $\Pi_r$  occurs. For any ergodic face  $\Lambda$ , not containing the pairs*

$$(l, k), \quad l \in \overline{1, b_{r-1}}, \quad k \in \overline{1, a_{r-1}}, \quad (36)$$

*the following holds true: for any pairs as*

$$(b, a_r), \quad b \in \overline{1, b_{r-1}}, \quad (37)$$

*belonging to  $\Lambda$ , the corresponding component of the vector field is negative :*

$$v_{b, a_r}^\Lambda < 0.$$

*If the case  $U_r$  occurs, then for any ergodic face  $\Lambda$ , not containing the pairs (36), the following components of the vector field are negative*

$$v_{b_r, a}^\Lambda < 0, \quad a \in \overline{1, a_{r-1}},$$

*under the condition, of course, that  $(b_r, a) \in \Lambda$ .*

*Proof of Proposition 5.* Remind the notation  $T_r = (b_r, \dots, 1 | 1, \dots, a_r)$ . As it was mentioned above, the connection between  $T_{r-1}$  and  $T_r$  can be of two kinds —  $\Pi_r$  or  $U_r$ , which we write schematically as

$$\begin{aligned} \Pi_r : \quad T_r &= T_{r-1} \cup (\emptyset | a_r) \\ U_r : \quad T_r &= T_{r-1} \cup (b_r | \emptyset) \end{aligned}$$

---

<sup>1</sup>CLC of the numbers  $x_1, \dots, x_n$  is  $\sum_i \alpha_i x_i$  for some numbers  $\alpha_i > 0, i = \overline{1, n}$  such that  $\sum_i \alpha_i = 1$ .

Consider only the case  $\Pi_r$ , as the case  $U_r$  is symmetric. It is necessary to prove that for any ergodic face  $\Lambda$ , which does not contain

$$(l, k), \quad l \in \overline{1, b_{r-1}}, \quad k \in \overline{1, a_{r-1}},$$

for any pairs  $(b, a_r) \in \Lambda$ , where  $b \in \overline{1, b_{r-1}}$ , the inequality

$$v_{b, a_r}^\Lambda < 0.$$

holds. Thus we mean the faces with

$$\overline{\Lambda} = (l_m, \dots, l_r, b_{r-1}, \dots, 1 \mid 1, \dots, a_{r-1}, \widehat{a_r}, k_{r+1}, \dots, k_n).$$

For such faces  $v_{b, a_r}^\Lambda = v_{a_r}^{(+)} - V^{\overline{\Lambda}}$ .

Consider now the case when the set  $k_{r+1}, \dots, k_n$  is not empty. As  $\overline{\Lambda}$  corresponds to ergodic group of particles, then by lemma 12  $v_{k_{r+1}}^{(+)} < V^{\overline{\Lambda}}$ . As  $a_r < k_{r+1}$ , then

$$v_{a_r}^{(+)} < v_{k_{r+1}}^{(+)} < V^{\overline{\Lambda}} \quad \Rightarrow \quad v_{a_r}^{(+)} - V^{\overline{\Lambda}} < 0.$$

The case when the set  $k_{r+1}, \dots, k_n$  is empty corresponds to

$$\overline{\Lambda} = (l_m, \dots, l_r, b_{r-1}, \dots, 1 \mid 1, \dots, a_{r-1}). \quad (38)$$

Case  $\Pi_r$  includes two possible subcases

$$V^{T_{r-1}} < 0, \quad v_{a_r}^{(+)} < V^{T_{r-1}} \quad (39)$$

$$V^{T_{r-1}} > 0, \quad v_{b_{r-1}+1}^{(-)} < V^{T_{r-1}} \quad (40)$$

Consider firstly (40). If the set  $l_m, \dots, l_r$  is not empty, then the subcase (40) contradicts the ergodicity assumption for (38), thus it is impossible. If the set  $l_m, \dots, l_r$  is empty, then  $\overline{\Lambda} = T_{r-1}$  and the assumption (40) means that  $V^{\overline{\Lambda}} = V^{T_{r-1}} > 0$ . As  $v_{a_r}^{(+)} < 0$ , we easily conclude that in this case

$$v_{b, a_r}^\Lambda = v_{a_r}^{(+)} - V^{\overline{\Lambda}} < 0.$$

Consider now (39). If the set  $l_m, \dots, l_r$  is not empty, then due to the ergodicity of the group (38), we have strict inequality  $V^{\overline{\Lambda}} > V^{T_{r-1}}$ . If the set  $l_m, \dots, l_r$  is empty, then  $\overline{\Lambda} = T_{r-1}$  and consequently  $V^{\overline{\Lambda}} = V^{T_{r-1}}$ . Finally we conclude that in the subsituation (39) always

$$V^{\overline{\Lambda}} \geq V^{T_{r-1}}.$$

From (39) we have

$$v_{a_r}^{(+)} < V^{T_{r-1}} \quad \Rightarrow \quad v_{a_r}^{(+)} < V^{T_{r-1}} \leq V^{\overline{\Lambda}}$$

and it follows that  $v_{b, a_r}^\Lambda = v_{a_r}^{(+)} - V^{\overline{\Lambda}} < 0$ .

This ends the proof.

*Proof of Proposition 4.* Assume the above algorithm produces the chain of groups (33). Let  $B(\Lambda_1), B(\Lambda_2), \dots, B(\Lambda_{fin})$  be the faces in  $R_+^N$ , corresponding to the chain  $T_1, T_2, \dots, T_{fin}$  via the rule (31). Denote  $F_1, F_2, \dots, F_{fin}$  the closures of these faces in  $R_+^N$ . That is in notation (35)

$$F_i = cl(B(\Lambda_i)) = \left\{ x \in R_+^N : \begin{aligned} & x_{j,k} \geq 0, \quad (j,k) \notin \{1, \dots, b_r\} \times \{1, \dots, a_r\}, \\ & x_{j,k} = 0, \quad (j,k) \in \{1, \dots, b_r\} \times \{1, \dots, a_r\} \end{aligned} \right\}.$$

It is clear that  $F_1 \supset F_2 \supset \dots \supset F_{fin}$ , and moreover  $\dim F_i > \dim F_{i+1}$ . More exactly,  $\dim F_r - \dim F_{r+1} = b_r$  or  $a_r$  in the case  $\Pi_r$  or  $U_r$  correspondingly.

Let  $\Gamma_x(t) = (\gamma_{j,k}(t))$ ,  $(j, k) \in I_- \times I_+$  be the coordinate description of the trajectory  $\Gamma_x$ . To prove that  $\Gamma_x(t') \in F_r$  one should check that  $\gamma_{j,k}(t') = 0$  for all  $(j, k) \in \{1, \dots, b_r\} \times \{1, \dots, a_r\}$ . The trajectory goes along ergodic faces.

1) Maximal ergodic face is  $\Lambda_0 = R_+^N$ . The vector field  $v^{\Lambda_0}$  on this face is such that  $v_{1,1}^{\Lambda_0} = v_1^{(+)} - v_1^{(-)} < 0$ . Note that also for any other ergodic face  $\Lambda$ , containing the pair  $(1, 1)$ , the component  $v_{1,1}^\Lambda$  will also be negative, as by (22)–(24) it can take only one of three following negative values

$$v_1^{(+)} - v_1^{(-)}, \quad v_1^{(+)} - V^\Lambda \quad \text{or} \quad V^\Lambda - v_1^{(-)}. \quad (41)$$

Thus for any initial point  $x$  there is  $t_1 \geq 0$  such that  $\gamma_{1,1}(t_1) = 0$ , and moreover,  $\gamma_{1,1}(t) = 0 \quad \forall t \geq t_1$ .

2) Thus  $\Gamma_x(t_1) \in F_1$ . If the case  $\Pi_2$  occurs, then we have to show the existence of  $t_2 \geq t_1$  such that  $\gamma_{1,2}(t) = 0 \quad \forall t \geq t_2$ . If it appeared that  $\Gamma_x(t_1) \in F_2$ , then just put  $t_2 = t_1$ . If however  $\Gamma_x(t_1) \notin F_2$ , that is  $\gamma_{1,2}(t_1) > 0$ , then  $\Gamma_x(t_1)$  belongs to some ergodic face  $\Lambda \ni (1, 2)$ . By proposition 5  $v_{1,2}^\Lambda < 0$ , and thus there is  $t_2 > t_1$  such that  $\gamma_{1,2}(t_2) = 0$  (that is  $\Gamma_x(t_2) \in F_2$ ). In future the dynamical system will never quit  $F_2$ . In fact, assume the contrary. Note that  $\Gamma_x(t_2)$  can belong either to  $\Lambda_2$ , or to its boundary (remind that  $\overline{\Lambda_2} = \{(1, 1), (1, 2)\}$  and  $cl(\Lambda_2) = F_2$ ). For the trajectory to quit  $F_2$  it is necessary that it used some outgoing ergodic face  $\Lambda'$ . There are two possibilities to do this. The first possibility is  $(1, 1) \in \Lambda'$ . But in this case (see (41))  $v_{1,1}^{\Lambda'} < 0$  and we get contradiction with the hypothesis that  $\Lambda'$  is an ergodic outgoing face. The second possibility is  $(1, 1) \notin \Lambda'$  and  $(1, 2) \in \Lambda'$ . But according to the proposition 5 for any such face  $v_{1,2}^{\Lambda'} < 0$ , and thus the dynamical system cannot quit  $F_2$  along such face  $\Lambda'$ . This gives the contradiction.

If the case  $U_2$  occurred then, quite similarly, one show existence of  $t_2 \geq t_1$  such that  $\gamma_{2,1}(t) = 0 \quad \forall t \geq t_2$ .

r) We can use further the induction, using subsequently proposition 5, to show on the step  $r$ , that there exists  $t_r \geq t_{r-1}$  such that for any  $t \geq t_r$

- $\gamma_{b,a_r}(t_r) = 0 \quad \forall b \in \overline{1, b_{r-1}}$ , if the case  $\Pi_r$  holds,
- $\gamma_{b_r,a}(t_r) = 0 \quad \forall a \in \overline{1, a_{r-1}}$ , if the case  $U_r$  holds.

Let us show now that in any case  $\Gamma_x(t) \in F_r$  for all  $t \geq t_r$ . For concreteness consider only the case  $\Pi_r$ , that is when

$$\begin{aligned} F_{r-1} &= \{x \in R_+^N : x_{i,j} = 0 \quad \forall (i, j) \in \{b_{r-1}, \dots, 1\} \times \{1, \dots, a_{r-1}\}\}, \\ F_r &= \{x \in R_+^N : x_{i,j} = 0 \quad \forall (i, j) \in \{b_{r-1}, \dots, 1\} \times \{1, \dots, a_r\}\}, \quad a_r = a_{r-1} + 1. \end{aligned}$$

Assume that the trajectory of the dynamical system  $\Gamma_x(t)$ , being at time  $t = t_r$  in  $F_r$ , will leave it at some future moment. The set  $F_r$  is a finite union of faces having various dimensions. One should understand then which outgoing ergodic faces  $\Lambda'$  can be used. Again there are two possibilities..

Case 1:  $\Lambda' \cap \{b_{r-1}, \dots, 1\} \times \{1, \dots, a_{r-1}\} = \emptyset$ , that is  $\Lambda' \subset F_{r-1}$ . Then there exists  $b \in \{b_{r-1}, \dots, 1\}$  such that  $(b, a_r) \in \Lambda'$  (otherwise  $\Lambda' \subset F_r$ , which gives the contradiction). By proposition 5 we have  $v_{b,a_r}^{\Lambda'} < 0$ . This contradicts to the fact that the face  $\Lambda'$  is outgoing.

Case 2:  $\Lambda' \cap \{b_{r-1}, \dots, 1\} \times \{1, \dots, a_{r-1}\} \neq \emptyset$ . Consider

$$q = \min \left\{ n : \Lambda' \cap \{b_n, \dots, 1\} \times \{1, \dots, a_n\} \neq \emptyset \right\}.$$

Assume for definiteness, that on step  $q$  of the algorithm we have

$$T_q = T_{q-1} \cup (b_q | \emptyset).$$

Then there exists such  $a \in \{1, \dots, a_{q-1}\}$ , that  $(b_q, a) \in \Lambda'$ . Applying Proposition 5, to  $\Lambda'$  we get  $v_{b_q, a}^{\Lambda'} < 0$  and come to the contradiction because  $\Lambda'$  is outgoing.

Thus there exists a time moment  $t_{fin} > 0$  such that for  $t \geq t_{fin}$  the trajectory hits the final ergodic face  $F_{fin}$ , which is the complement to the final group (32).

Important remark is that the sequence of times

$$t_1 \leq t_2 \leq \dots \leq \dots \leq t_r \leq \dots \leq t_{fin}$$

depends on the initial point. In particular, for some initial points some consequent moments  $t_{r-1}$  and  $t_r$  can coincide.

**Remark 10** Consider the following modification of the algorithm: in cases **2a)** and **r-a)** change the conditions  $V^{T_1} < 0$  and  $V^{T_{r-1}} < 0$  on  $V^{T_1} \leq 0$  and  $V^{T_{r-1}} \leq 0$  correspondingly. All the rest we leave untouched. It is easy to see that all results of this section hold after such modification as well. In particular, our study covers the situation when<sup>2</sup>  $V^{T_{fin}} = 0$ .

From the above it follows that any trajectory  $\Gamma_x(t)$  reaches the final face in finite time. To proceed with the proof of Theorem 11 we will prove the following lemma.

**Lemma 11** *For any initial point  $x$  the path  $\Gamma_x(t)$  has finite number of transitions from one face to another, until it reaches one of the final faces. In other words the sequence of faces, passed by the path  $\Gamma_x(t)$ , is finite and the last element of this sequence is the final face.*

*Proof of Lemma 11.* Consider an arbitrary trajectory  $\Gamma_x(t)$ . Let  $\{\Lambda_i^x\}$  be a sequence of all faces visited by this trajectory. Denote  $\{T_i^x\}$  the sequence of the corresponding groups, where  $T_i^x = \overline{\Lambda_i^x}$ . We want to show that the sequence  $\{\Lambda_i^x\}$  is finite.

Two cases are possible for the transition  $\Lambda_i^x \rightarrow \Lambda_{i+1}^x$ , or equivalently, for the transition  $T_i^x \rightarrow T_{i+1}^x$ . If the face  $\Lambda_i^x$  is ergodic, then the group  $T_{i+1}^x$  is obtained by adding some new particle type to the group  $T_i^x$ . During this transition the dimension of  $\Lambda_i^x$  decreases. If the face  $\Lambda_i^x$  is non-ergodic, then  $\Lambda_{i+1}^x$  is the minimal outgoing face, containing  $\Lambda_i^x$  (see lemma 3). In the transition  $\Lambda_i^x \rightarrow \Lambda_{i+1}^x$  from  $T_i^x$  some types are deleted, and the dimension of  $\Lambda_i^x$  increases. Thus, the transition  $T_i^x \rightarrow T_{i+1}^x$  can occur with two operations: adding some new type and deleting some types. The same type can be added and deleted several times. If we could show that addition and deletion are possible only finite number of times, that will give finiteness of the sequence  $\{\Lambda_i^x\}$ .

Note the following fact. Take for example some (+)-type  $k$ . Then it can be deleted from the group on some step if and only if on the previous step we added to the group some (+)-type with smaller number (that is with greater velocity). That is why the type 1, plus or minus, can be added only once and cannot be deleted. (+)-type 2 can be deleted only after adding (+)-type 1. Similarly for (-)-type 2. That is why type 2, plus or minus, can be added to the group not more than twice and can be deleted not more than once. One can prove by induction that any type can be deleted and added not more that finite number of times.

*Proof of Theorem 3.*

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<sup>2</sup>  $V^{T_{fin}}$  coincides with the asymptotic boundary velocity of our system (see Subsection 5.4).

Let the chain (33) be the result of the algorithm. Three cases are possible, defined by simple inequalities between  $v_L^{(-)}$ ,  $v_K^{(+)}$  and  $V^{T_{fin}}$ .

$v_K^{(+)} < V^{T_{fin}} < v_L^{(-)}$ . This corresponds to part 1 of lemma 9, that is  $\Lambda_{fin} = \overline{T_{fin}} = \{0\}$ . Thus (Proposition 4), all trajectories of the dynamical system  $U_t$  reach 0 for finite time and finite number of changes. Note that from this, using well-known methods (see [1, 4]), one can get alternative proof of ergodicity of  $D(t)$ , in addition to the one of theorem 2. The first assertion of theorem 3 is proved.

$V^{T_{fin}} < v_K^{(+)} < 0$ . This case corresponds to part 2 of lemma 9, and thus,  $T_{fin} = (L, \dots, 1 \mid 1, \dots, K_1)$ , where  $K_1 < K$ . From the rules of the algorithm it follows immediately that  $v_{K_1+1}^{(+)} > V^{T_{fin}}$ , but  $v_{K_1}^{(+)} < V^{T_{fin}}$ . Thus (see theorem 2), the process  $D_{\overline{T_{fin}}}(t)$  is ergodic, and the face  $\Lambda_{fin} = \overline{T_{fin}}$  is also ergodic. Find now the vector  $v^{\Lambda_{fin}}$ . Note that

$$\Lambda_{fin} = \mathcal{L}(L, K_1) = \{(i, k) : i = 1, \dots, L, k = K_1 + 1, \dots, K\}.$$

To find components of  $v_{i,k}^{\Lambda_{fin}}$  we use the formulas (22)–(24)

$$v_{i,k}^{\Lambda_{fin}} = v_k^{(+)} - V^{\overline{\Lambda_{fin}}} > v_{K_1+1}^{(+)} - V^{T_{fin}} > 0 \quad \forall (i, j) \in \{1, \dots, L\} \times \{K_1 + 1, \dots, K\}, \quad (42)$$

$$v_{i,k}^{\Lambda_{fin}} = 0 \quad \forall (i, j) \in \{1, \dots, L\} \times \{1, \dots, K_1\}.$$

By Proposition 4 any trajectory, in finite time and after finite number of changes, will reach  $\mathcal{L}(L, K_1)$ , and will move along it with constant speed  $v^{\Lambda_{fin}}$ , having strictly positive components (42). By standard methods of [1, 4], we conclude that  $D(t)$  is transient. *The second assertion of theorem 3 is proved.*

$0 < v_L^{(-)} < V^{T_{fin}}$ . This case corresponds to part 3 of lemma 9, and the proof is completely similar to the previous case. That proves assertion 3 of theorem 3.

The fourth assertion of theorem 3 is a corollary of proposition 4 and lemma 11.

Theorem 3 is proved.

## 5.4 Proof of theorem 1

If associated random walk  $D(t)$  is ergodic, then by lemma 5 the speed of the boundary equals  $V$  which is defined by (4).

Let the process  $D(t)$  be non-ergodic. Then there are two possible cases:  $v_K^{(+)} > V$  or  $v_L^{(-)} < V$ . From the previous Subsection 5.3 it follows that any trajectory  $\Gamma_x(t)$  reaches the final face in finite time and during this time only finite number of changing the face occurs.

The following assertion is an obvious analog of the proposition 1.4.3 of [1].

**Lemma 12** *For any  $t \geq 0$  and any initial point  $x$*

$$\frac{D_{xM}(tM)}{M} \rightarrow \Gamma_x(t)$$

*a.e. as  $M \rightarrow \infty$ .*

Let  $v_K^{(+)} > V$ . We have proved that any trajectory of the dynamical system  $U^t$  reaches the final face  $\mathcal{L}(L, K_1)$ , where the coordinates of the induced vector are positive. By lemma 12 the coordinates  $d_{q,r}(t)$  of the process  $D(t)$ , where  $q = 1, \dots, L$ ,  $r = K_1 + 1, \dots, K$ , grow linearly (a.e.) as  $t \in \infty$ . In other words (+)-types with numbers  $r = K_1 + 1, \dots, K$  fall behind the boundary and do not contribute to its velocity. It means that the boundary velocity is defined only by the particles of types  $q = 1, \dots, L$ ,  $r = 1, \dots, K_1$  and are given by formula (4). The case of  $v_L^{(-)} < V$  is quite similar.

## 6 Appendix

### 6.1 Proof of Lemma 1

Let the face  $\Lambda$  be such that  $\bar{\Lambda}$  is not the direct product. Put

$$\begin{aligned} I_-^{\bar{\Lambda}} &= \{i \in I_- : \exists k \in I_+, (i, k) \in \bar{\Lambda}\} \\ I_+^{\bar{\Lambda}} &= \{k \in I_+ : \exists i \in I_-, (i, k) \in \bar{\Lambda}\} \end{aligned}$$

Choose an “appropriate” face  $\Lambda_0$  so that  $\bar{\Lambda}_0 = I_-^{\bar{\Lambda}} \times I_+^{\bar{\Lambda}}$ . To prove the lemma it is sufficient to show

$$\mathcal{D} \cap \Lambda = \mathcal{D} \cap \Lambda_0$$

As  $\Lambda \supset \Lambda_0$ , we always have  $\mathcal{D} \cap \Lambda \supset \mathcal{D} \cap \Lambda_0$ . Let us prove that  $\mathcal{D} \cap \Lambda \subseteq \mathcal{D} \cap \Lambda_0$ . Let  $(i, k) \in \bar{\Lambda}_0$  and  $(i, k) \notin \bar{\Lambda}$ . Then there exist  $m \in I_+$  and  $n \in I_-$  such that  $(i, m) \in \bar{\Lambda}$ ,  $(n, k) \in \bar{\Lambda}$  and the following equation holds

$$d_{i,k}(t) + d_{n,m}(t) = d_{i,m}(t) + d_{n,k}(t)$$

Take arbitrary element  $d = (d_{j,l})$  of the set  $\mathcal{D} \cap \Lambda$ . As its coordinates  $d_{i,m}(t) = d_{n,k}(t) = 0$ , then  $d_{i,k} = 0$  for all  $(i, k) \in \bar{\Lambda}_0$ . Thus,  $d \in \mathcal{D} \cap \Lambda_0$ , and the lemma is proved.

### 6.2 Technical lemma

For shortness denote

$$f(k) = V(I_-, \{1, \dots, k\}), \quad g(l) = V(\{1, \dots, l\}, I_+)$$

**Lemma 13** *We have*

- $v_{k+1}^{(+)} < f(k+1) \iff f(k+1) < f(k), k = 1, \dots, K-1,$
- $v_{k+1}^{(+)} > f(k+1) \iff f(k+1) > f(k), k = 1, \dots, K-1$
- $v_k^{(+)} > f(k) \implies v_{k+1}^{(+)} > f(k+1), k = 2, \dots, K-1$

*Similarly,*

- $v_{l+1}^{(-)} < g(l+1) \iff g(l+1) < g(l), l = 1, \dots, L-1$
- $v_{l+1}^{(-)} > g(l+1) \iff g(l+1) > g(l), l = 1, \dots, L-1$
- $v_l^{(-)} < g(l) \implies v_{l+1}^{(-)} < g(l+1), l = 2, \dots, L-1$

Proof. We prove the first three items. The others are quite similar. Using (4) one can check

$$f(k+1) = \alpha f(k+1) + \beta f(k+1) = \alpha f(k) + \beta v_{k+1}^{(+)}$$

for some  $\alpha, \beta > 0$  such that  $\alpha + \beta = 1$ . It follows that

$$\alpha(f(k+1) - f(k)) = \beta(v_{k+1}^{(+)} - f(k+1))$$



Thus,  $v_{k+1}^{(+)} < f(k+1) \iff f(k+1) < f(k)$ . If  $v_k^{(+)} > f(k)$ , using  $v_k^{(+)} < v_{k+1}^{(+)}$ , we get

$$f(k+1) < \alpha v_k^{(+)} + \beta v_{k+1}^{(+)} < \alpha v_{k+1}^{(+)} + \beta v_{k+1}^{(+)} = v_{k+1}^{(+)}$$

Lemma is proved.

Let  $K_1$  and  $L_1$  be defined by (7) and (6). It follows from the lemma that

$$f(1) > \dots > f(K_1) < f(K_1 + 1) < \dots < f(K)$$

and

$$g(1) < \dots < g(L_1) > g(L_1 + 1) > \dots > g(L)$$

So the minimum of  $f(k)$  is reached at point  $K_1$  and maximum of  $g(l)$  is reached at point  $L_1$ .

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